Exact solution for mean first-passage time on a pseudofractal scale-free web

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The explicit determinations of the mean first-passage time (MFPT) for trapping problem are limited to some simple structure, e.g., regular lattices and regular geometrical fractals, and determining MFPT for random walks on other media, especially complex real networks, is a theoretical challenge. In this paper, we investigate a simple random walk on the the pseudofractal scale-free web (PSFW) with a perfect trap located at a node with the highest degree, which simultaneously exhibits the remarkable scale-free and small-world properties observed in real networks. We obtain the exact solution for the MFPT that is calculated through the recurrence relations derived from the structure of PSFW. The rigorous solution exhibits that the MFPT approximately increases as a power-law function of the number of nodes, with the exponent less than 1. We confirm the closed-form solution by direct numerical calculations. We show that the structure of PSFW can improve the efficiency of transport by diffusion, compared with some other structure, such as regular lattices, Sierpinski fractals, and T-graph. The analytical method can be applied to other deterministic networks, making the accurate computation of MFPT possible.

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I. INTRODUCTION

The problem of random walks (diffusion) is a major theme and understanding its behavior is central to a wide range of applications [1, 2, 3, 4]. Among various interesting issues of random walks, trapping plays a significant role in an increasing number of disciplines, e.g., physics [5, 6], society [7], computer [8], and biology [9], to name a few. The classic trapping issue first introduced in [10] is a random-walk problem, in which a trap is placed at a given location, absorbing all walkers that visit it.

An important quantity related to trapping problem is the trapping time (first-passage time, survival time, or the mean walk length) or the mean time to absorption. The trapping time of a given site s is the expected time for a walker starting from s to first reach the trap. This quantity is useful in the study of transport-limited reactions [11, 12], target search [13, 14] and other physical problems. The average trapping time, also known as the mean first-passage time (MFPT), characterizes the process of trapping. It is defined as the average of survival times over all starting sites.

In the past few decades, there has been considerable interest in computing the mean first-passage time, in order to obtain the dependence of this primary quantity on the system size or other parameters. In a seminal work, using the approach based on generating functions, Montroll derived the rigorous results for MFPT of random walks on regular lattices with a variety of dimensions [10]. Recently, by applying a decimation procedure, Kozak and Balakrishnan obtained the accurate solutions for MFPT on a family of Sierpinski fractals [15, 16]; using an analogous but different method, Agliari got the exact expression for the MFPT for a random walker on T-fractal [17]. In spite of these rigorous results, the explicit determination of MFPT for random walks with a trap on other media is still open [18].

It is well established that the scaling of MFPT is related to the underlying structural properties of the media in which the walkers are confined [1, 2, 3, 4]. Extensive empirical studies [19, 20] have revealed that most real networked systems share some striking features, such as scale-free behavior [21] and small-world effects [22]. These newly-found properties have a profound effect on almost all dynamical processes taking place on the networks [23, 24, 25], including disease spreading [26], games [27], synchronization [28], and so on. Very recently, a lot of activities have been devoted to the study of influences of scale-free and small-world characteristics on the behavior of random walks, uncovering many unusual and exotic phenomena about random walks [29, 30, 31, 32, 33, 34, 35, 36, 37]. However, to best of our knowledge, rigorous solution for MFPT of trapping problem on scale-free small-world networks is missing.

In this paper, we study the classic trapping problem on a deterministic network, called pseudofractal scale-free web (PSFW) [38]. We focus on a peculiar case with the trap fixed at a hub node (node with the highest degree). The PSFW is a very useful toy model that captures simultaneously scale-free small-world properties, thus provides a good facility to investigate analytically trapping process upon it. We derive an exact formula for the mean first-passage time characterizing the trapping pro-

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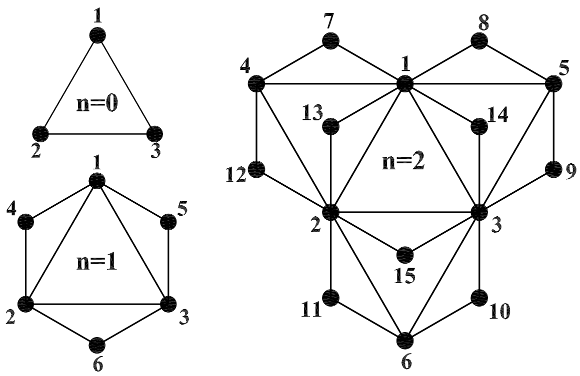

FIG. 1: The first three generations of the pseudofractal scale-free web.

cess. The analytic approach is based on an algebraic iterative procedure obtained from the particular construction of PSFW. The obtained rigorous result shows that the MFPT grows as a power-law function of the number of network nodes with the exponent less than 1, which implies that in contrast to regular lattices, Sierpinski fractals, and T-graph, the PSFW tends to speed up the diffusion process. Our study opens the way to theoretically investigate the MFPT of a random walker on a wide range of deterministic networks [39, 40, 41, 42].

II. THE PSEUDOFRACTAL SCALE-FREE WEB

Here we introduce the pseudofractal scale-free web defined in a recursive way [38], which has attracted an amount of attention [43, 44, 45, 46]. We investigate the PSFW model because of its intrinsic interest and its deterministic construction, which allows one to study analytically their topological properties and dynamical processes on it.

The pseudofractal scale-free web, denoted by G_n after n ($n \geq 0$) generation evolution, is constructed as follows. For $n = 0$, G_0 is a triangle of edges connecting three nodes (vertices, sites). For $n \geq 1$, G_n is obtained from G_{n-1}: every existing edge in G_{n-1} introduces a new node connected to both ends of the edge. Figure 1 illustrates the construction process for the first three generations.

According to the network construction, one can see that at each step n_i ($n_i \geq 1$) the number of newly introduced nodes is $L(n_i) = 3^{n_i}$. From this result, we can easily compute the network order (i.e., the total number of nodes) V_n at step n:

$$V_n = \sum_{n_i=0}^{n} L(n_i) = \frac{3^{n+1}+3}{2}. \tag{1}$$

Let $k_i(n)$ be the degree of a node i at time n, which entered the network at step n_i ($n_i \geq 0$). Then

$$k_i(n) = 2^{n-n_i+1}. \tag{2}$$

From Eq. (2), one can easily see that at each step the degree of a node doubles, i.e.,

$$k_i(n) = 2\, k_i(n-1). \tag{3}$$

The PSFW presents some typical characteristics of real-life networks in nature and society. It has a power-law degree distribution $P(k) \sim k^{-\gamma}$ with the exponent $\gamma = 1 + \frac{\ln 3}{\ln 2}$ [38]. Its average path length (APL), defined as the mean of shortest distance between all pairs of nodes, increases logarithmically with network order [38, 46]. In the large network order limit, the average clustering coefficient tends to $\frac{4}{5}$. Thus, the PSFW exhibits small-world behavior [22]. In addition, its node degree correlations are negative, the average degree of nearest neighbors, $k_{\rm nn}(k)$, for nodes with degree k is approximately a power-law function of k with a negative exponent $-(2 - \frac{\ln 3}{\ln 2})$ [44].

While the graph introduced above visually looks very similar to a fractal (see Fig. 1), the similarities are only superficial. The distinction is obvious. Fractals have a finite dimension (i.e., their APL grows as a power of the vertex number) [47]. In sharp contrast, the network under consideration has an infinite dimension (i.e., its APL increases slower than any power of the network order) [48]. This is why it is called *pseudofractal scale-free web*.

After introducing the PSFW, in what follows we will study the mean first-passage time for random walks with a single immobile trap on the web, by applying a method similar to but different from that introduced in [15, 16].

III. FORMULATION OF THE PROBLEM

In this section we formulate the problem of a simple unbiased Markovian random walk of a particle on PSFW G_n in the presence of a trap or a perfect absorber located on a given node. In order to distinguish different nodes, we label all the nodes belonging to G_n in the following way. The initial three nodes in G_0 are labeled as 1, 2, and 3, respectively. In each new generation, only the new nodes created at this generation are labeled, while the labels of all old nodes remain unchanged, i.e., we label new nodes as $M+1, M+2, \ldots, M+\Delta M$, where M is the total number of the pre-existing nodes and ΔM is the number of newly-created nodes. Eventually, every node is labeled by a unique integer, at time n all nodes are labeled from 1 to $V_n = \frac{3^{n+1}+3}{2}$, see Fig. 1.

We locate the trap at node 1 [49], denoted as i_T. Note that the particular selection we made for the trap location makes the analytical computation process (that will be shown in detail in the next section) easily iterated as we can identify the trap node i_T since the first generation. At each time step (taken to be unity), the particle, starting from any node except the trap i_T, moves to any of its nearest neighbors with equal probability. It is easily seen that in the presence of the trap i_T fixed on node

1, the walker (particle) will be inevitably absorbed [32]. This random walk process can be described by a specifying the set of transition probabilities W_{ij} for the particle of going from node i to node j. In fact, all the entries constitute a matrix $\mathbf{W}$ that is a (V_n-1)-order sub-matrix of $(\mathbf{Z}^{-1})\mathbf{A}$ with the first row and column corresponding to trap being removed, where $\mathbf{Z}$ and $\mathbf{A}$ are separately the adjacency matrix and diagonal degree matrix of G_n [50].

Let T_i be survival time for a walker initially placed at node i to first reach the trap i_T. Then, the set of this interesting quantity obeys the following recurrence equation [51]

$$T_i = \sum_j W_{ij}\, T_j + 1, \tag{4}$$

where $i \neq i_T$. Eq. (4) expresses the Markovian property of the random walk, it may be recast in matrix notation as

$$\mathbf{T} = \mathbf{W}\,\mathbf{T} + \mathbf{e}, \tag{5}$$

where $\mathbf{T} = (T_2, T_3, \cdots, T_{V_n})^\top$ (the superscript $\top$ of the vector represents transpose) is a (V_n-1)-dimensional vector, $\mathbf{e}$ is the (V_n-1)-dimensional unit vector $(1, 1, \cdots, 1)^\top$, and $\mathbf{W}$ is the transition matrix. From Eq. (5), we can easily obtain

$$\mathbf{T} = \mathbf{L}\,\mathbf{e}, \tag{6}$$

where

$$\mathbf{L} = (\mathbf{I} - \mathbf{W})^{-1}, \tag{7}$$

in which $\mathbf{I}$ is an identity matrix with an order $(V_n-1) \times (V_n-1)$. Equation (7) is the fundamental matrix of the Markov chain representing the unbiased random walk. Actually, the matrix $\mathbf{I}-\mathbf{W}$ in Eq. (7) is the normalized discrete Laplacian matrix of G_n whose first row and column that correspond to the trap node have been deleted.

Then, the mean first-passage time (MFPT), or the average of the mean time to absorption, $\langle T\rangle_n$, which is the average of T_i over all initial nodes distributed uniformly over nodes in G_n other than the trap, is given by

$$\langle T\rangle_n = \frac{1}{V_n-1}\sum_{i=2}^{V_n} T_i = \frac{1}{V_n-1}\sum_{i=2}^{V_n}\sum_{j=2}^{V_n} L_{ij}. \tag{8}$$

Equation (8) can be easily explained from the Markov chain representing the random walk. In fact, the entry L_{ij} of the fundamental matrix $\mathbf{L}$ for the Markov process denotes the expected number of times that the process is in the transient state j, being started in the transient state i.

The quantity of MFPT is very important since it measures the efficiency of the trapping process: the smaller the MFPT, the higher the efficiency, and vice versa. Equation (8) shows that the problem of calculating $\langle T\rangle_n$ is reduced to finding the sum of all elements of matrix $\mathbf{L}$. Notice that the order of $\mathbf{L}$ is $(V_n-1)\times(V_n-1)$, where V_n increases exponentially with n, as shown in Eq. (1). So, for large n, it becomes difficult to obtain $\langle T\rangle_n$ through direct calculation from Eq. (8), one can compute directly the MFPT only for the first several generations, see Table I. However, the recursive construction of PSFW allows one to compute analytically MFPT to achieve a closed-form solution, the derivation details of which will be given in next section.

TABLE I: The average of the mean time to absorption obtained by direct calculation from Eq. (8). Since for large networks, the computation of the MFPT from Eq. (8) is prohibitively time and memory consuming, we calculate the MFPT for only the first several generations.

n	V_n	$\langle T\rangle_n$
0	3	4/2
1	6	19/5
2	15	101/14
3	42	571/41
4	123	3329/122
5	366	19699/365
6	1095	117401/1094
7	3282	702091/3281
8	9843	4205729/9842

IV. EXACT SOLUTION FOR MEAN FIRST-PASSAGE TIME

Before deriving the general formula for MFPT, $\langle T\rangle_n$, we first establish the scaling relation dominating the evolution of T_i^n with generation n, where T_i^n is the trapping time for a walk originating at node i on the nth generation of PSFW.

A. Evolution scaling for trapping time

We begin by recording the numerical values of T_i^n. Obviously, for all $n \geq 0$, $T_1^n = 0$; for $n = 0$, it is a trivial case, we have $T_2^0 = T_3^0 = 2$. For $n \geq 1$, the values of T_i^n can be obtained straightforwardly via Eq. (8). In the generation $n = 1$, by symmetry we have $T_2^1 = T_3^1 = 4$, $T_4^1 = T_5^1 = 3$, and $T_6^1 = 5$. Analogously, for $n = 2$, the solutions are $T_2^2 = T_3^2 = 8$, $T_4^2 = T_5^2 = 6$, $T_6^2 = 10$, $T_7^2 = T_8^2 = 4$, $T_9^2 = T_{12}^2 = 8$, $T_{10}^2 = T_{11}^2 = 10$, $T_{13}^2 = T_{14}^2 = 5$, and $T_{15}^2 = 9$. Table II lists the numerical values of T_i^n for some nodes up to $n = 6$.

The numerical values listed in Table II show that for a given node i we have $T_i^{n+1} = 2\,T_i^n$. That is to say, upon growth of PSFW from n to generation $n+1$, the mean time to first reach the trap doubles. For example, $T_2^6 = 2\,T_2^5 = 4\,T_2^4 = 8\,T_2^3 = 16\,T_2^2 = 32\,T_2^1 = 64\,T_2^0 =$

TABLE II: Mean time to absorption T_i^n for a random walker starting from node i on PSFW for various n. Notice that owing to the obvious symmetry, nodes in a parenthesis are equivalent, since they have the same trapping time. All the values are calculated straightforwardly from Eq. (8).

$n\backslash i$	(2,3)	(4,5)	(6)	(7,8)	(9,12)	(10,11)	(13,14)	(15)
0	2							
1	4	3	5					
2	8	6	10	4	8	10	5	9
3	16	12	20	8	16	20	10	18
4	32	24	40	16	32	40	20	36
5	64	48	80	32	64	80	40	72
6	128	96	160	64	128	160	80	144

128, $T_4^6 = 2\,T_4^5 = 4\,T_4^4 = 8\,T_4^3 = 16\,T_4^2 = 32\,T_4^1 = 96$, $T_7^6 = 2\,T_7^5 = 4\,T_7^4 = 8\,T_7^3 = 16\,T_7^2 = 64$, and so on. This is a basic character of random walks on the PSFW, which can be established from the arguments below.

Consider an arbitrary node i in the PSFW G_n after n generation evolution of the network. From Eq. (2), we know that upon growth of PSFW to generation $n+1$, the degree k_i of node i doubles. Let the mean transmit time for going from node i to any of its k_i old neighbors be Y; and let the mean transmit time for going from any of its k_i new neighbors to one of the k_i old neighbors be Z. Then we can establish the following underlying backward equations

$$\begin{cases} Y = \frac{1}{2} + \frac{1}{2}(1+Z), \\ Z = \frac{1}{2} + \frac{1}{2}(1+Y), \end{cases} \tag{9}$$

which leads to $Y = 2$. That is to say, the passage time from any node i ($i \in G_n$) to any node j ($j \in G_n$) increases by a factor of 2, upon the network growth from generation n to generation $n+1$. Thus, we have $T_i^{n+1} = 2\,T_i^n$, which will be useful for deriving the formula for the mean first-passage time in the following text.

B. Formula for the mean first-passage time

Having obtained the scaling of mean transmit time for old nodes, we now determine the average of the mean time to absorption, aiming to derive an exact solution. We represent the set of nodes in G_n as Λ_n, and denote the set of nodes created at generation n by $\overline{\Lambda}_n$. Thus we have $\Lambda_n = \overline{\Lambda}_n \cup \Lambda_{n-1}$. For the convenience of computation, we define the following quantities for $m \leq n$:

$$T_{m,\mathrm{total}}^n = \sum_{i \in \Lambda_m} T_i^n, \tag{10}$$

and

$$\overline{T}_{m,\mathrm{total}}^n = \sum_{i \in \overline{\Lambda}_m} T_i^n. \tag{11}$$

Then, we have

$$T_{n,\mathrm{total}}^n = T_{n-1,\mathrm{total}}^n + \overline{T}_{n,\mathrm{total}}^n. \tag{12}$$

Next we will explicitly determine the quantity $T_{n,\mathrm{total}}^n$. To this end, we should firstly determine $\overline{T}_{n,\mathrm{total}}^n$.

We examine the mean time to absorption for the first several generations of PSFW. In the case of $n = 1$, by construction of the PSFW, it follows that $T_4^1 = \frac{1}{2}(1+T_1^1) + \frac{1}{2}(1+T_2^1)$, $T_5^1 = \frac{1}{2}(1+T_1^1) + \frac{1}{2}(1+T_3^1)$, and $T_6^1 = \frac{1}{2}(1+T_2^1) + \frac{1}{2}(1+T_3^1)$. Thus,

$$\begin{aligned} \overline{T}_{1,\mathrm{total}}^1 &= \sum_{i \in \overline{\Lambda}_1} T_i^1 = T_4^1 + T_5^1 + T_6^1 \\ &= 3 + (T_1^1 + T_2^1 + T_3^1) = 3 + \overline{T}_{0,\mathrm{total}}^1. \end{aligned} \tag{13}$$

Similarly, for $n = 2$ case, $T_7^2 = \frac{1}{2}(1+T_1^2) + \frac{1}{2}(1+T_4^2)$, $T_8^2 = \frac{1}{2}(1+T_1^2) + \frac{1}{2}(1+T_5^2)$, $T_9^2 = \frac{1}{2}(1+T_3^2) + \frac{1}{2}(1+T_5^2)$, $T_{10}^2 = \frac{1}{2}(1+T_3^2) + \frac{1}{2}(1+T_6^2)$, $T_{11}^2 = \frac{1}{2}(1+T_2^2) + \frac{1}{2}(1+T_6^2)$, $T_{12}^2 = \frac{1}{2}(1+T_2^2) + \frac{1}{2}(1+T_4^2)$, $T_{13}^2 = \frac{1}{2}(1+T_1^2) + \frac{1}{2}(1+T_2^2)$, $T_{14}^2 = \frac{1}{2}(1+T_1^2) + \frac{1}{2}(1+T_3^2)$, and $T_{15}^2 = \frac{1}{2}(1+T_2^2) + \frac{1}{2}(1+T_3^2)$, so that

$$\begin{aligned} \overline{T}_{2,\mathrm{total}}^2 &= \sum_{i \in \overline{\Lambda}_2} T_i^2 = \sum_{i=7}^{15} T_i^2 \\ &= 3^2 + 2\,(T_1^2 + T_2^2 + T_3^2) + (T_4^2 + T_5^2 + T_6^2) \\ &= 3^2 + 2\,\overline{T}_{0,\mathrm{total}}^2 + \overline{T}_{1,\mathrm{total}}^2. \end{aligned} \tag{14}$$

Proceeding analogously, it is not difficult to derive that

$$\begin{aligned} \overline{T}_{n,\mathrm{total}}^n = 3^n &+ \overline{T}_{n-1,\mathrm{total}}^n + 2\,\overline{T}_{n-2,\mathrm{total}}^n + \dots \\ &+ 2^{n-2}\,\overline{T}_{1,\mathrm{total}}^n + 2^{n-1}\,\overline{T}_{0,\mathrm{total}}^n, \end{aligned} \tag{15}$$

and

$$\begin{aligned} \overline{T}_{n+1,\mathrm{total}}^{n+1} = 3^{n+1} &+ \overline{T}_{n,\mathrm{total}}^{n+1} + 2\,\overline{T}_{n-1,\mathrm{total}}^{n+1} + \dots \\ &+ 2^{n-1}\,\overline{T}_{1,\mathrm{total}}^{n+1} + 2^n\,\overline{T}_{0,\mathrm{total}}^{n+1}, \end{aligned} \tag{16}$$

where 3^n and 3^{n+1} are indeed the numbers of nodes generated at generations n and $n+1$, respectively. Equation (16) minus Eq. (15) times 4 and making use of the relation $T_i^{n+1} = 2\,T_i^n$, one gets

$$\overline{T}_{n+1,\mathrm{total}}^{n+1} - 3^{n+1} = \overline{T}_{n,\mathrm{total}}^{n+1} + 4\left(\overline{T}_{n,\mathrm{total}}^n - 3^n\right), \tag{17}$$

which may be rewritten as

$$\overline{T}_{n+1,\mathrm{total}}^{n+1} = 6\,\overline{T}_{n,\mathrm{total}}^n - 3^n. \tag{18}$$

Using $\overline{T}_{1,\mathrm{total}}^1 = 11$, Eq. (18) is solved inductively

$$\overline{T}_{n,\mathrm{total}}^n = \frac{5}{3} \times 6^n + 3^{n-1}. \tag{19}$$

Substituting Eq. (19) for $\overline{T}^{n}_{n,\text{total}}$ into Eq. (12), we have

$$\begin{aligned} T^{n}_{n,\text{total}} &= T^{n}_{n-1,\text{total}} + \frac{5}{3}\times 6^{n} + 3^{n-1} \\ &= 2\,T^{n-1}_{n-1,\text{total}} + \frac{5}{3}\times 6^{n} + 3^{n-1}\,. \end{aligned} \tag{20}$$

Considering the initial condition $T^{0}_{0,\text{total}} = 4$, Eq. (20) is resolved by induction to yield

$$T^{n}_{n,\text{total}} = \frac{5}{2}\times 6^{n} + 3^{n} + 2^{n-1}\,. \tag{21}$$

Plugging the last expression into Eq. (8), we arrive at the accurate formula for the average of the mean time to absorption at the trap located at node 1 on the nth of the pseudofractal scale-free web:

$$\begin{aligned} \langle T\rangle_n &= \frac{1}{V_n-1}\sum_{i=2}^{V_n} T_i = \frac{1}{V_n-1} T^{n}_{n,\text{total}} \\ &= \frac{5\times 6^{n} + 2\times 3^{n} + 2^{n}}{3^{n+1}+1}\,. \end{aligned} \tag{22}$$

We have checked our analytic formula against numerical values quoted in Table I. For the range of $0 \le n \le 8$, the values obtained from Eq. (22) completely agree with those numerical results on the basis of the direct calculation through Eq. (8). This agreement serves as an independent test of our theoretical formula.

We continue to show how to represent MFPT as a function of network order, with the aim of obtaining the scaling between these two quantities. Recalling Eq. (1), we have $3^{n+1} = 2V_n - 3$ and $n+1 = \log_3(2V_n-3)$. Hence, Eq. (22) can be recast as

$$\langle T\rangle_n = \frac{\frac{5}{6}(2V_n-3)^{1+\frac{\ln 2}{\ln 3}} + \frac{2}{3}(2V_n-3) + \frac{1}{2}(2V_n-3)^{\frac{\ln 2}{\ln 3}}}{2V_n-2}. \tag{23}$$

For large network, i.e., $V_n \to \infty$,

$$\langle T\rangle_n \sim (V_n)^{\frac{\ln 2}{\ln 3}}, \tag{24}$$

where the exponent $\frac{\ln 2}{\ln 3} < 1$. Thus, in the large limit of network order V_n, the MFPT increases algebraically with increasing order of the network.

The above scaling of the MFPT with network order is different from those previously obtained for other media. For example, on regular lattices with large order N, the leading behavior of MFPT $\langle T\rangle$ is $\langle T\rangle \sim N^2$, $\langle T\rangle \sim N\ln N$, and $\langle T\rangle \sim N$ for dimensions $d=1$, $d=2$, and $d=3$, respectively [10]. Again for instance, on planar Sierpinski gasket [15] and Sierpinski tower [16] in 3 Euclidean dimensions, the asymptotic behavior scales separately as $\langle T\rangle \sim N^{1.464}$ and $\langle T\rangle \sim N^{1.293}$. At last, on T-fractal, the leading asymptotic scaling behaves as $\langle T\rangle \sim N^{1.631}$ [17]. Thus, in contrast to regular lattices, Sierpinski fractals, and T-fractal, the trapping process on the pseudofractal scale-free web is more efficient. It is expected that the efficiency of trapping process on stochastic scale-free networks is also high, since they have similar structural features as the PSFW.

Why is the MFPT for the PFSW far smaller than that for other lattices? We speculate that the heterogeneity of the pseudofractal scale-free web is responsible for this distinction, which may be seen from the following heuristic argument. In the PFSW, there are a few nodes with large degree that are connected to most nodes in the web, which results in the logarithmic scaling of the average path length with network order [38, 46]. A walker starting from some node will arrive at 'large' nodes with ease. Since 'large' nodes, including the trap node, are linked to one another, so the walker can find the trap in a short time. While for other regular lattices, they are almost homogeneous, and their average path lengths are much larger than that of the PFSW. This leads to a long MFPT. It should be noted that we only give a possible explanation for the shorter MFPT on the PFSW, the genuine reason for this deserves further study in the future.

V. CONCLUSIONS

In summary, we have investigated the classic trapping problem on a deterministically growing network, named pseudofractal scale-free web (PSFW) that can reproduce some remarkable properties of various real-life networks, such as scale-free feature and small-world behavior, and thus can mimic some real systems to some extent (to what extent it does is still an open question). With the help of recursion relations derived from the structure of PSFW, we have obtained the solution for the MFPT for random walks on PSFW, with a trap fixed at a hub node. The exact result shows that the MFPT increases as a power-law function of network order with the exponent less than 1, which is in contrast to the well-known previously obtained results that for regular lattices, Sierpinski fractals, and T-graph with order N, their MFPT $\langle T\rangle$ behaves as $\langle T\rangle \sim N^{\alpha}$ with $\alpha > 1$. Therefore, the structure of PSFW has a profound impact on the trapping problem on it. To the best of our knowledge, our result may be the first exact scaling about MFPT for random walks on scale-free small-world networks.

We should stress that although we have only computed the MFPT for a particular deterministic network, our analytical technique could guide and shed light on related studies for other deterministic networks by providing a paradigm for calculating the MFPT. Moreover, since exact solutions can serve for a guide to and a test of approximate solutions or numerical simulations, we also believe our accurate closed-form solutions can prompt related studies on stochastic networks. At last, as future work, it is interesting to compute higher moments of trapping time for the PSFW and compare the scaling with that of homogeneous fractal lattices [52]. Another future job of interest is to study the case when the trap is mobile instead of being fixed at one of the hub nodes of the web.

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